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The analytical solution of the Riemann problem in relativistic hydrodynamics

By JOSÉ Mª MARTÍ† AND EWALD MÜLLER

Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Str. 1, 8046 Garching b. München, Germany

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We consider the decay of an initial discontinuity in a polytropic gas in a Minkowski space–time (the special relativistic Riemann problem). In order to get a general analytical solution for this problem, we analyse the properties of the relativistic flow across shock waves and rarefactions. As in classical hydrodynamics, the solution of the Riemann problem is found by solving an implicit algebraic equation which gives the pressure in the intermediate states. The solution presented here contains as a particular case the special relativistic shock-tube problem in which the gas is initially at rest. Finally, we discuss the impact of this result on the development of high-resolution shock-capturing numerical codes to solve the equations of relativistic hydrodynamics.

1. Introduction

In this paper we present the general analytical solution of the Riemann problem for a polytropic gas in a Minkowski space–time. As in the Newtonian case (see e.g. Courant & Friedrichs 1948) the special relativistic Riemann problem consists of calculating the one-dimensional gas flow resulting from the decay of a discontinuity which separates two constant initial states L (left) and R (right).

In Newtonian hydrodynamics the Riemann problem has played a very important role both by providing analytical solutions against which hydrodynamical codes can be tested (see e.g. Sod 1978), and in the development of the codes themselves. In fact, most of the modern so-called high-resolution shock-capturing (HRSC) techniques, which have been developed during the last decade, are frequently based on the exact or approximate solution of Riemann problems between adjacent numerical cells (see e.g. the recent book by LeVeque 1991). These HRSC techniques are all based on the pioneering work of Godunov (1959), who first used the exact solution of Riemann problems to construct a hydrodynamical code.

Prior to this work the analytical solution of the special relativistic Riemann problem was only known for a special case commonly called the *shock-tube problem*, where the fluid of both constant initial states is at rest (Thompson 1986). This fact has limited its use for testing special relativistic hydrodynamical codes. However, the general analytical solution derived by us (see below) opens the door for the use of exact Riemann solvers in *numerical relativistic hydrodynamics*.

In the past elementary nonlinear special relativistic waves have been extensively treated in the literature (for a review, see e.g. the book by Anile 1989). However, none of these investigations aimed at solving the general Riemann problem. Nevertheless, let

† On leave from Departamento de Física Teórica, Universidad de Valencia, 46100 Burjassot (Valencia), Spain.

us mention some of the most interesting ones. The theory of relativistic simple waves and shocks was established by Taub (1948). Later the jump conditions and adiabats were further studied by Israel (1960), Lichnerowicz (1967, 1970, 1971) and Thorne (1973). Some analytical results concerning self-similar solutions of relativistic blast waves were obtained by Johnson & McKee (1971), Eltgroth (1971, 1972), Blandford & McKee (1976) and Bogoyavlenski (1978). Finally, the process of shock formation by steepening of simple waves has been studied in detail by Liang (1977) and Muscato (1988) for relativistic fluid dynamics and relativistic magneto fluid dynamics, respectively.

Let us now summarize the main features of the time evolution of the Riemann problem. The decay of the initial discontinuity qualitatively occurs in the same way both in relativistic and Newtonian hydrodynamics giving rise, in general, to three elementary nonlinear waves (see e.g. Landau & Lifshitz 1987). Two of them can be shocks or rarefaction waves, one moving towards the initial left state, and the other towards the initial right state. Between them, two new states appear, namely L_* and R_*, separated from each other by the third wave, which is a contact discontinuity moving with the fluid. Across the contact discontinuity pressure and velocity are constant, while the density exhibits a jump. Accordingly, the time evolution of a Riemann problem can be represented as

$$I \rightarrow L \mathscr{W}_{\leftarrow} L_* \mathscr{C} R_* \mathscr{W}_{\rightarrow} R, \tag{1}$$

where $\mathscr{W}$ and $\mathscr{C}$ denote a simple wave (shock or rarefaction) and a contact discontinuity, respectively. The arrows $(\leftarrow | \rightarrow)$ indicate the direction (left | right) from which fluid elements enter the corresponding wave.

As in the Newtonian case, the compressible character of shock waves (density and pressure rise across the shock; see Taub 1978) allows us to discriminate between shocks ($\mathscr{S}$) and rarefaction waves($\mathscr{R}$):

$$\mathscr{W}_{\leftarrow(\rightarrow)} = \begin{cases} \mathscr{R}_{\leftarrow(\rightarrow)}, & p_b \leqslant p_a \\ \mathscr{S}_{\leftarrow(\rightarrow)}, & p_b > p_a, \end{cases} \tag{2}$$

where p is the pressure and subscripts a and b denote quantities ahead of and behind the wave. For the Riemann problem $a \equiv L(R)$ and $b \equiv L_*(R_*)$ for $\mathscr{W}_{\leftarrow}$ and $\mathscr{W}_{\rightarrow}$, respectively.

In complete analogy with the classical case, the possible types of decay of an initial discontinuity can be reduced to

$$(a) \qquad I \rightarrow L \mathscr{S}_{\leftarrow} L_* \mathscr{C} R_* \mathscr{S}_{\rightarrow} R, \quad p_L < p_{L_*} = p_{R_*} > p_R, \tag{3}$$

$$(b) \qquad I \rightarrow L \mathscr{S}_{\leftarrow} L_* \mathscr{C} R_* \mathscr{R}_{\rightarrow} R, \quad p_L < p_{L_*} = p_{R_*} \leqslant p_R, \tag{4}$$

$$(c) \qquad I \rightarrow L \mathscr{R}_{\leftarrow} L_* \mathscr{C} R_* \mathscr{R}_{\rightarrow} R, \quad p_L \geqslant p_{L_*} = p_{R_*} \leqslant p_R. \tag{5}$$

The standard shock-tube problem, considered by Thompson (1986), belongs to case (*b*), whereas case (*a*) represents two colliding fluids with a large relative velocity and case (*c*) represents two fluids moving apart from each other.

The solution of the Riemann problem consists in finding the intermediate states, L_* and R_*, as well as the positions of the waves separating the four states (which only depend on L, L_*, R_* and R).

The condition of self-similar flow through rarefaction waves and the Rankine–Hugoniot relations across shocks allow us to connect the intermediate states I_* $(I = L, R)$ with their corresponding initial state I. The analytical solution of the Riemann problem in classical hydrodynamics (see e.g. Courant & Friedrichs 1948) rests on the fact that the velocity in the intermediate states, v_{I_*}, can be written as a function of the pressure p_{I_*}. Thus, once p_{I_*} is known, v_{I_*} and all other unknown state quantities of I_* can be calculated. In order to obtain the pressure p_{I_*} one uses the jump conditions across the contact discontinuity, which are given by

$$p_{L_*} = p_{R_*}(= p_*), \tag{6}$$

$$v_{L_*}(p_*) = v_{R_*}(p_*). \tag{7}$$

Equation (7) is an implicit algebraic equation in p_* and can be solved by means of an iterative method. The function $v_{I_*}(p)$ is constructed by using the relations across the corresponding wave which are given by (2).

In order to solve the Riemann problem in relativistic hydrodynamics we shall follow the same procedure. First, in §2 we give the equations of relativistic hydrodynamics in one spatial dimension. Then in §§3 and 4 we review the properties of the flow across rarefaction waves and shocks, respectively, and give the expressions relating velocity and pressure behind the waves to the flow conditions in the state ahead of the waves. In §5, the results of the two previous sections are combined to solve the Riemann problem. Finally, in §6 we discuss the applicability of the solution in numerical relativistic hydrodynamics.

Throughout the paper the Newtonian limit of the relativistic expressions is given whenever it is of interest.

After this work was finished, the authors became aware of the paper by Smoller & Temple (1993), in which the general Riemann problem for relativistic hydrodynamics is solved for $\gamma = 1$.

2. The equations of relativistic hydrodynamics in one spatial dimension

Let us consider a perfect fluid described by a four-velocity vector field u^μ and an energy momentum tensor

$$T^{\mu\nu} = \rho h u^\mu u^\nu + p\eta^{\mu\nu} \quad (\mu, \nu = 0, \ldots, 3), \tag{8}$$

where ρ is the proper rest-mass density, p is the pressure and h is the specific enthalpy defined by

$$h = 1 + \epsilon + p/\rho, \tag{9}$$

with ϵ being the specific internal energy. Throughout this paper, units in which the speed of light is set to unity are used. In Cartesian coordinates $x^\mu = (t, x, y, z)$ the Minkowski metric tensor $\eta^{\mu\nu}$ is given by

$$\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1). \tag{10}$$

The motion of the fluid is governed by the equation of continuity

$$(\rho u^\mu)_{,\mu} = 0, \tag{11}$$

and the conservation of energy-momentum

$$T^{\mu\nu}_{,\nu} = 0. \tag{12}$$

In (11) and (12) we have used the summation convention and $f_{,\nu} \equiv \partial f/\partial x^\nu$.

For a perfect fluid, the energy conservation equation $T^{0\nu}_{,\nu} = 0$ is consistent with the adiabatic flow condition

$$s_{,\mu} u^\mu = 0 \tag{13}$$

where s is the specific entropy.

The system of equations (11) and (12) with $\mu = 0, \ldots, 3$, is closed by means of the normalization condition for the four-velocity

$$u^\mu u_\mu = -1 \tag{14}$$

and an equation of state (EOS) which we shall assume as given in the form

$$p = p(\rho, \epsilon). \tag{15}$$

In the following we restrict our discussion to a polytropic EOS

$$p = K\rho^\gamma, \tag{16}$$

K and γ, the adiabatic index ($1 < \gamma < 2$), being two constants for which the specific internal energy is given (up to an additive constant) by

$$\epsilon = \frac{p}{(\gamma - 1)\rho}. \tag{17}$$

We further only consider motion in the x-direction, i.e. one-dimensional flows. Consequently, u^μ is given by

$$u^\mu = W(1, v, 0, 0), \tag{18}$$

where v is the spatial velocity

$$v = \mathrm{d}x/\mathrm{d}t \tag{19}$$

and

$$W = \frac{1}{(1 - v^2)^{\frac{1}{2}}} \tag{20}$$

the corresponding Lorentz factor.

With these restrictions the relativistic hydrodynamical equations can be written as

$$\frac{\partial D}{\partial t} + \frac{\partial Dv}{\partial x} = 0, \tag{21}$$

$$\frac{\partial S}{\partial t} + \frac{\partial (Sv + p)}{\partial x} = 0, \tag{22}$$

$$\frac{\partial \tau}{\partial t} + \frac{\partial (S - Dv)}{\partial x} = 0, \tag{23}$$

where we have introduced the definitions

relativistic rest-mass density: $D = \rho u^0 = \rho W,$ (24)

momentum density: $S = T^{01} = \rho h W^2 v,$ (25)

energy density: $\tau = T^{00} - \rho u^0 = \rho h W^2 - p - \rho W.$ (26)

In the limit $v \to 0, h \to 1$ the conserved quantities D, S and τ approach their Newtonian counterparts ρ, ρv and $\rho E = \rho\epsilon + \frac{1}{2}\rho v^2$, which obey the classical conservation equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0, \tag{27}$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial (\rho v^2 + p)}{\partial x} = 0, \tag{28}$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial v(\rho E + p)}{\partial x} = 0. \tag{29}$$

Note that in the Newtonian limit first-order terms in the relativistic energy equation – see (23) and (26) – cancel, while the second-order terms give rise to (29).

3. Relation between the flow velocity and pressure behind relativistic rarefaction waves

Rarefaction waves are simple waves in which the pressure and the density of a gas element decrease when crossing them. They are self-similar solutions of the flow equations, i.e. all quantities describing the fluid depend on x and t only through the combination $\xi = x/t$. As in the Newtonian case, the conservation of entropy along fluid lines forces self-similar flows to be isentropic, which can easily be seen by writing (13) in terms of the similarity variable ξ

$$(v-\xi)\frac{\mathrm{d}s}{\mathrm{d}\xi} = 0$$

and considering that, in general, $v \neq \xi$.

The flow equations reduce to the following condition for the differentials of ρ and v:

$$W^2\,\mathrm{d}v \pm \frac{c_s}{\rho}\,\mathrm{d}\rho = 0, \tag{30}$$

the plus and minus signs corresponding to rarefaction waves propagating to the left ($\mathscr{R}_{\leftarrow}$) and right ($\mathscr{R}_{\rightarrow}$), respectively. The sound speed, $c_s = [(1/h)(\partial p/\partial \rho)\mid_s]^{\frac{1}{2}}$, is also given by

$$c_s = \pm\frac{v-\xi}{1-v\xi} \tag{31}$$

(where the $\pm$ sign has the same meaning as in (30)) or, equivalently,

$$\xi = \frac{v \mp c_s}{1 \mp v c_s}. \tag{32}$$

Equation (30) states that the Riemann invariant J_+ (J_-)

$$J_{\pm} = \tfrac{1}{2}\ln\left(\frac{1+v}{1-v}\right) \pm \int \frac{c_s}{\rho}\,\mathrm{d}\rho \tag{33}$$

is constant through rarefaction waves propagating to the left (right) (see Taub 1948).

For a polytropic EOS like (16) the sound speed can be written as a function of ρ only:

$$c_s^2 = \frac{K\gamma(\gamma-1)\,\rho^{(\gamma-1)}}{(\gamma-1)+K\gamma\rho^{(\gamma-1)}} \tag{34}$$

and the integral in (33) can be evaluated analytically giving

$$\tfrac{1}{2}\ln\left(\frac{1+v}{1-v}\right) \pm \frac{1}{(\gamma-1)^{\frac{1}{2}}}\ln\left[\frac{(\gamma-1)^{\frac{1}{2}}+c_s}{(\gamma-1)^{\frac{1}{2}}-c_s}\right] = \text{constant} \tag{35}$$

or

$$\left(\frac{1+v}{1-v}\right)\left[\frac{(\gamma-1)^{\frac{1}{2}}+c_s}{(\gamma-1)^{\frac{1}{2}}-c_s}\right]^{\pm 2(\gamma-1)^{-\frac{1}{2}}} = \text{constant}. \tag{36}$$

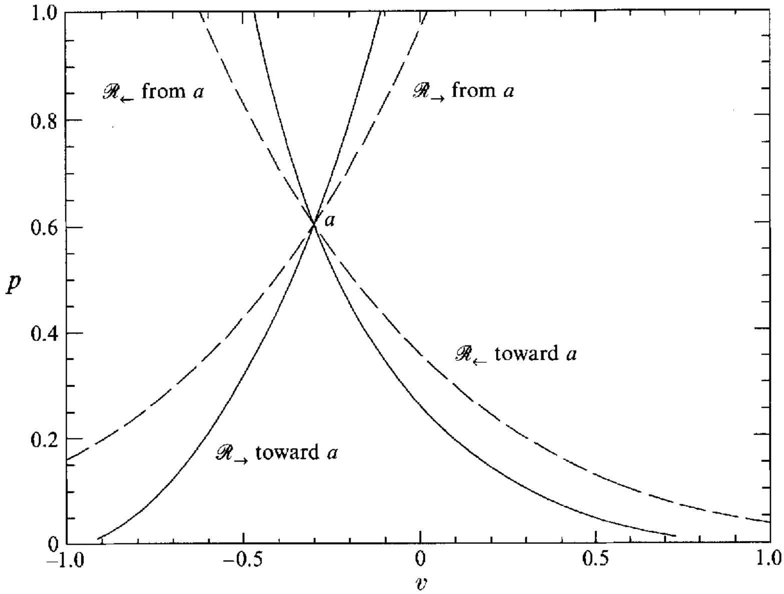

FIGURE 1. Loci of states which can be connected with a given state a by means of relativistic rarefaction waves propagating to the left ($\mathcal{R}_{\leftarrow}$) and to the right ($\mathcal{R}_{\rightarrow}$) and moving towards or away from a. The corresponding Newtonian curves (dashed lines) are shown for comparison. The state a is characterized by $p_a = 0.6$, $\rho_a = 1.0$, and $v_a = -0.3$ (pressure and density are given in arbitrary units but compatible with our choice of the speed of light being equal to 1). A polytropic equation of state with $\gamma = \frac{5}{3}$ was assumed.

The Newtonian limit of (36) leads to the well-known condition for rarefaction waves in classical hydrodynamics (see e.g. Courant & Friedrichs 1948):

$$v \pm \frac{2}{\gamma - 1} c_s = \text{constant}. \tag{37}$$

Expression (36) can be used to connect two states inside the rarefaction wave†, and in particular the states ahead of and behind the wave, a and b. Considering that in a Riemann problem the state a is known, (36) allows one to obtain a relation between the flow velocity and pressure behind a rarefaction wave:

$$v_b = \frac{(1+v_a) A_{\pm}(p_b) - (1-v_a)}{(1+v_a) A_{\pm}(p_b) + (1-v_a)}, \tag{38}$$

where $A_{\pm}$ is defined by

$$A_{\pm}(p) = \left[\frac{(\gamma-1)^{\frac{1}{2}} - c_s(p)}{(\gamma-1)^{\frac{1}{2}} + c_s(p)} \frac{(\gamma-1)^{\frac{1}{2}} + c_s(p_a)}{(\gamma-1)^{\frac{1}{2}} - c_s(p_a)}\right]^{\pm 2(\gamma-1)^{-\frac{1}{2}}}. \tag{39}$$

Equation (38) can be stated in a more compact way:

$$v_b = \mathcal{R}^a_{\rightleftharpoons}(p_b). \tag{40}$$

The function $\mathcal{R}^a_{\rightleftharpoons}(p)$ is shown in figure 1, the various branches of the curves corresponding to rarefaction waves propagating towards or away from a. Rarefaction waves move towards (away from) a if the pressure inside the rarefaction is smaller

† We will use this fact in §5 to construct the analytical solution inside the rarefaction.

(larger) than p_a. The symmetry of the function with respect to the axis $v = v_a$, characteristic of Newtonian rarefactions, is lost (it only remains for the special case $v_a = 0$) owing to the finite light speed which limits the flow velocity.

In a Riemann problem the state a is ahead of the wave and only those branches corresponding to waves propagating towards a in figure 1 must be considered. Moreover, one can discriminate between waves propagating towards the left and right by taking into account that the initial left (right) state can only be reached by a wave propagating towards the left (right).

4. Relation between post-shock flow velocity and pressure for relativistic shock waves

The Rankine–Hugoniot conditions relate the states on both sides of the shock and are based on the continuity of the mass flux and the energy–momentum flux across shocks. Their relativistic version was first obtained by Taub (1948).

If Σ is a hypersurface in Minkowski space across which ρ, u^μ and $T^{\mu\nu}$ are discontinuous, the relativistic Rankine–Hugoniot conditions are given by

$$[\rho u^\mu]\, n_\mu = 0, \tag{41}$$

$$[T^{\mu\nu}]\, n_\nu = 0, \tag{42}$$

where n_μ is the unit normal to Σ and we have used the notation

$$[F] = F_a - F_b, \tag{43}$$

F_a and F_b being the boundary values of F on the two sides of Σ.

In a frame in which the shock is moving with coordinate velocity V_s along the x-axis, n_μ is given by

$$n_\mu = W_s(-V_s, 1, 0, 0), \tag{44}$$

where W_s is the shock's Lorentz factor

$$W_s = \frac{1}{(1-V_s^2)^{\frac{1}{2}}}. \tag{45}$$

Equation (41) allows one to introduce the (invariant) mass flux across the shock, j,

$$j \equiv W_s D_a(V_s - v_a) = W_s D_b(V_s - v_b). \tag{46}$$

According to our definition, j is positive for shocks propagating towards the right. Note that our convention differs from that of Landau & Lifshitz (1987) and that of Courant & Friedrichs (1948) but follows Taub (1978) and Anile (1989).

Now, the Rankine–Hugoniot conditions ((41) and (42) with $\mu = 0, 1$) can be written, in terms of the conserved quantities D, S and τ and the mass flux j, as

$$[v] = -\frac{j}{W_s}\left[\frac{1}{D}\right], \tag{47}$$

$$[p] = \frac{j}{W_s}\left[\frac{S}{D}\right], \tag{48}$$

$$[vp] = \frac{j}{W_s}\left[\frac{\tau}{D}\right], \tag{49}$$

where in the derivation of (49) we have made use of the relation

$$S = v(\tau + D + p). \tag{50}$$

The Newtonian limit of (47)–(49) gives the well-known classical Rankine–Hugoniot relations

$$[v] = -j_N[1/\rho], \tag{51}$$

$$[p] = j_N[v], \tag{52}$$

$$[vp] = j_N[E], \tag{53}$$

where the Newtonian mass flux, j_N, is defined by

$$j_N \equiv \rho_a(V_s - v_a) = \rho_b(V_s - v_b). \tag{54}$$

For a Riemann problem in Newtonian hydrodynamics, the pre-shock state a is known. Thus, (51)–(53) and the EOS can be used to determine the post-shock state b, i.e. the quantities p_b, ρ_b, ϵ_b, v_b and j_N (or V_s). In fact, (52) relates the velocity and pressure jumps to the mass flux. Hence, j_N can be expressed in terms of the known pre-shock state variables and the post-shock pressure p_b, which then allows one to obtain the post-shock velocity v_b as a function of p_b. In order to solve the relativistic Riemann problem we shall proceed along the same steps as in the Newtonian case.

4.1. *Post-shock flow velocity v_b as a function of p_b, j and V_s*

Taking into account (50), one can rewrite (48) in the form

$$[p] = \frac{j}{W_s}\left([v] + \left[\frac{\tau v}{D}\right] + \left[\frac{pv}{D}\right]\right). \tag{55}$$

As the next step one writes the terms τ_b/D_b and $1/D_b$ in (55) as functions of v_b, p_b, j and V_s using (49) and (47), respectively. Finally, after some rearrangement, one finds

$$v_b = \left[v_a - \frac{W_s(p_a - p_b)}{j} + v_a\frac{\tau_a + p_a}{D_a}\right]\left[1 + \frac{\tau_a + p_b}{D_a} + \frac{W_s v_a(p_b - p_a)}{j}\right]^{-1}, \tag{56}$$

or in terms of the state variables p, ρ, h and v

$$v_b = \left[h_a W_a v_a + \frac{W_s(p_b - p_a)}{j}\right]\left[h_a W_a + (p_b - p_a)\left(\frac{W_s v_a}{j} + \frac{1}{\rho_a W_a}\right)\right]^{-1}, \tag{57}$$

which has the correct Newtonian limit

$$v_b = v_a + \frac{p_b - p_a}{j_N}. \tag{58}$$

4.2. *Mass flux and shock velocity as functions of the post-shock pressure*

Starting from (46) one expresses V_s in terms of the mass flux (see the Appendix), i.e. the problem is reduced to writing the mass flux in terms of the post-shock pressure. In the Newtonian case one proceeds as follows. Firstly, (51) and (52) are combined to yield

$$j_N^2 = \frac{-[p]}{[1/\rho]}. \tag{59}$$

Next, ρ_b is eliminated using the EOS and the *Hugoniot adiabat*

$$[\epsilon + p/\rho] = \frac{1}{2}\left(\frac{1}{\rho_a} + \frac{1}{\rho_b}\right)[p], \tag{60}$$

which relates only thermodynamical quantities on both sides of the shock wave.

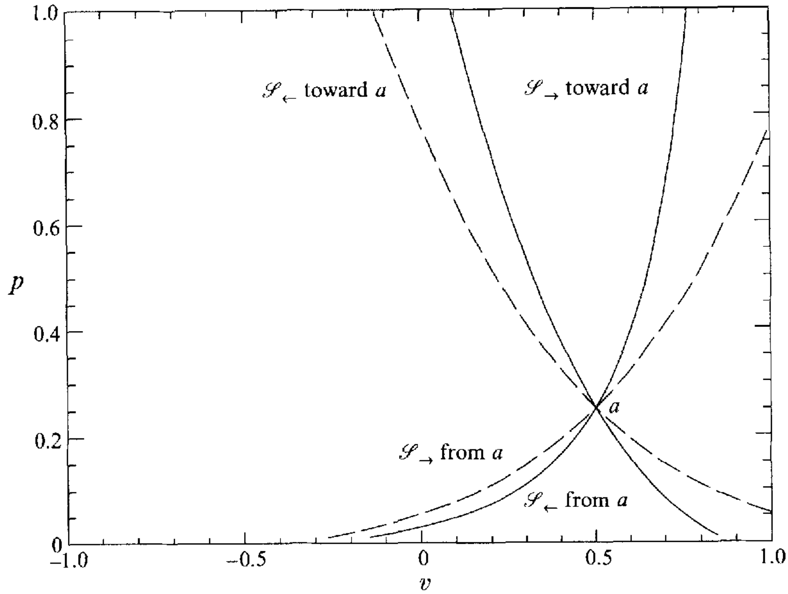

FIGURE 2. Loci of states which can be connected with a given state a by means of relativistic shock waves propagating to the left ($\mathscr{S}_{\leftarrow}$) and to the right ($\mathscr{S}_{\rightarrow}$) and moving towards or away from a. The corresponding Newtonian curves (dashed lines) are shown for comparison. The state a is characterized by $p_a = 0.25$, $\rho_a = 1.0$, and $v_a = -0.5$. A polytropic equation of state with $\gamma = \frac{5}{3}$ was assumed.

In relativistic hydrodynamics the analogous expression to (59) is obtained by multiplying (42) by n_μ and using the definition of the relativistic mass flux (see (46)), which leads to

$$j^2 = \frac{-[p]}{[h/\rho]}. \tag{61}$$

The *Taub adiabat* (Thorne 1973), the relativistic version of the Hugoniot adiabat, is obtained by multiplying (42) by $(hu_\mu)_a$ and by $(hu_\mu)_b$ and summing both resulting expressions (see, for example, Taub 1978). After a little algebra one finds

$$[h^2] = \left(\frac{h_b}{\rho_b} + \frac{h_a}{\rho_a}\right)[p]. \tag{62}$$

Next the post-shock density ρ_b is eliminated in both (61) and (62) using the relation

$$\rho_b = \frac{\gamma p_b}{(\gamma-1)(h_b-1)}, \tag{63}$$

which is valid for a polytropic EOS. Then the Taub adiabat can be rewritten in the form

$$h_b^2\left(1+\frac{(\gamma-1)(p_a-p_b)}{\gamma p_b}\right) - \frac{(\gamma-1)(p_a-p_b)}{\gamma p_b} h_b + \frac{h_a(p_a-p_b)}{\rho_a} - h_a^2 = 0, \tag{64}$$

which is a quadratic equation for the post-shock enthalpy h_b as a function of p_b. Considering further that both

$$1 > \frac{\gamma-1}{\gamma}\frac{p_b-p_a}{p_b} \tag{65}$$

and

$$h_a > \frac{p_a - p_b}{\rho_a} \tag{66}$$

hold independently whether $(p_a - p_b)$ is larger or smaller than zero, it can be shown that one of the two roots of (64) is always negative and must be discarded as a physical solution. Inserting the physical solution of (64) into the modified (61) (i.e. ρ_b being eliminated; see above) one obtains an expression that gives the square of the mass flux j^2 as a function of p_b. Finally, using the positive (negative) root of j^2 for shock waves propagating towards the right (left), (56) (or (57)) allows one to obtain the desired relation between the post-shock velocity v_b and the post-shock pressure p_b. In a compact way the relation reads

$$v_b = \mathscr{S}^a_{\rightleftharpoons}(p_b). \tag{67}$$

The function $\mathscr{S}^a_{\rightleftharpoons}(p)$ is shown in figure 2. Its various branches correspond to shock waves propagating towards or away from a. As in the case of rarefactions (see §3), the symmetry of the function $\mathscr{S}^a_{\rightleftharpoons}(p)$ with respect to the axis $v = v_a$, characteristic of its Newtonian analogue, is lost. Only in the special case $v_a = 0$ does the symmetry still exist. In order to select the relevant branch of the function $\mathscr{S}^a_{\rightleftharpoons}(p)$ (see figure 2) the same arguments as in the case of rarefaction waves can be used (see §3).

5. The solution of the Riemann problem in relativistic hydrodynamics

All states which can be connected to a given state R on the right through a wave propagating to the right can be represented in the (v, p)-plane by a curve $\mathscr{W}^R_{\rightarrow}$. The analytical expression for this curve is given by

$$\mathscr{W}^R_{\rightarrow} = \begin{cases} v = \mathscr{R}^R_{\rightarrow}, & p \leqslant p_R \\ v = \mathscr{S}^R_{\rightarrow}, & p > p_R. \end{cases} \tag{68}$$

Similarly, all points representing states connected to a given state L on the left by a wave propagating to the left lie on the curve

$$\mathscr{W}^L_{\leftarrow} = \begin{cases} v = \mathscr{R}^L_{\leftarrow}, & p \leqslant p_L \\ v = \mathscr{S}^L_{\leftarrow}, & p > p_L. \end{cases} \tag{69}$$

In the case of a Riemann problem, where the states L and R are the corresponding initial left and right states, the functions $\mathscr{W}^R_{\rightarrow}$ and $\mathscr{W}^L_{\leftarrow}$ allow one to determine the functions $v_{R_*}(p)$ and $v_{L_*}(p)$, respectively. The pressure p_* and the flow velocity v_* in the intermediate states are then given by the condition

$$v_{R_*}(p) = v_{L_*}(p). \tag{70}$$

Figures 3 and 4 show the solution of two particular Riemann problems. Once p_* and v_* are known, the remaining quantities can be derived: ρ_{L_*}, ρ_{R_*}, ϵ_{L_*} and ϵ_{R_*}, the (constant) propagation velocities of the waves (i.e. the shock velocities and/or the velocities of the head and tail of the rarefactions), and if rarefaction waves are produced by the decay of the initial discontinuity, the flow conditions inside them.

The EOS can be used to obtain the specific internal energy in the intermediate states once the density of these states is known. The remaining state variables of the intermediate state I_* can be calculated using the relations between I_* and the respective initial state I, which are given through the corresponding wave.

In the case of shock waves (63) and (64) can be used to compute ρ_{I_*}. For this purpose

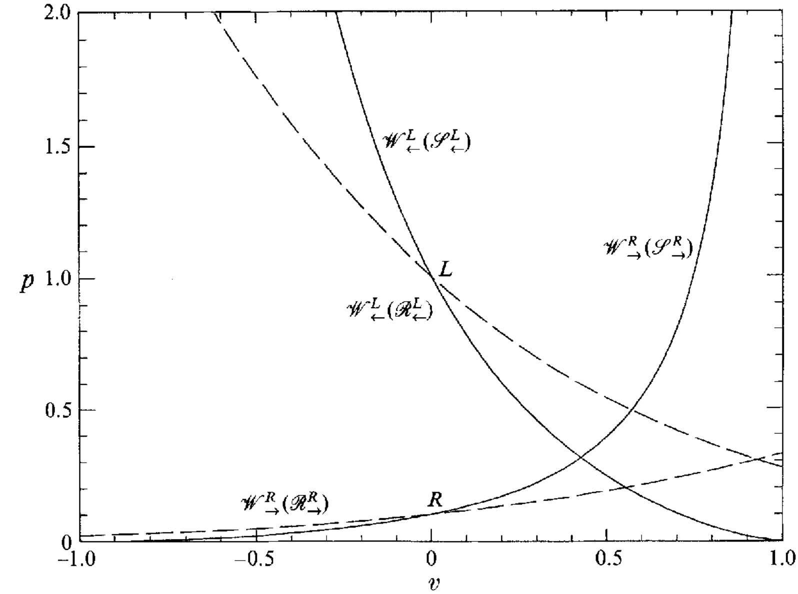

FIGURE 3. Graphical solution in the (p, v)-plane of the relativistic Riemann problem with initial data $p_L = 1.0$, $\rho_L = 1.0$, $v_L = 0$; $p_R = 0.1$, $\rho_R = 0.125$ and $v_R = 0$. A polytropic equation of state with $\gamma = \frac{5}{3}$ was assumed. This problem is called *Sod's problem* (Sod 1978). The crossing point of the two solid lines gives the pressure and the flow velocity in the intermediate states. The corresponding Newtonian solution (dashed lines) is shown for comparison.

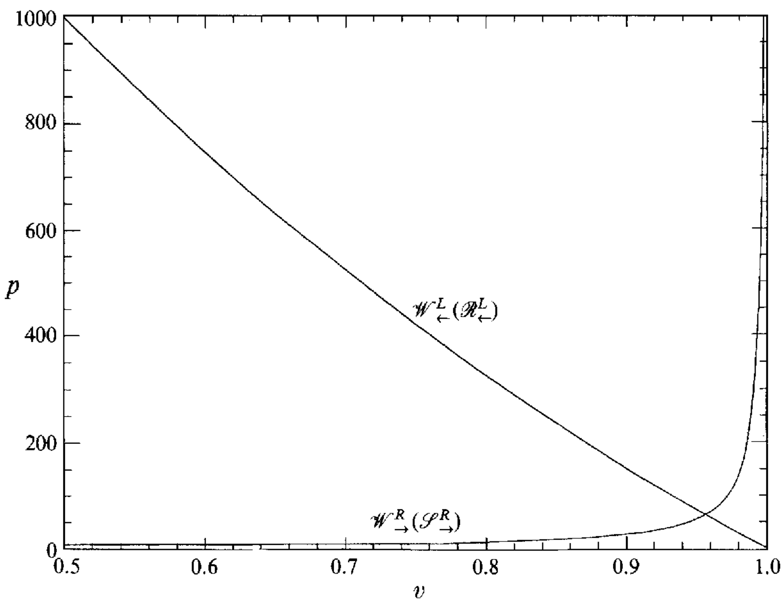

FIGURE 4. Graphical solution in the (p, v)-plane of the relativistic Riemann problem with initial data $p_L = 10^3$, $\rho_L = 1.0$, $v_L = 0.5$; $p_R = 1.0$, $\rho_R = 1.0$ and $v_R = 0$. A polytropic equation of state with $\gamma = \frac{5}{3}$ was assumed. The decay of the discontinuity leads to a flow velocity in the intermediate states close to the speed of light. Note the asymptotic behaviour of $\mathscr{W}^R_{\rightarrow}$ when it approaches $v = 1$.

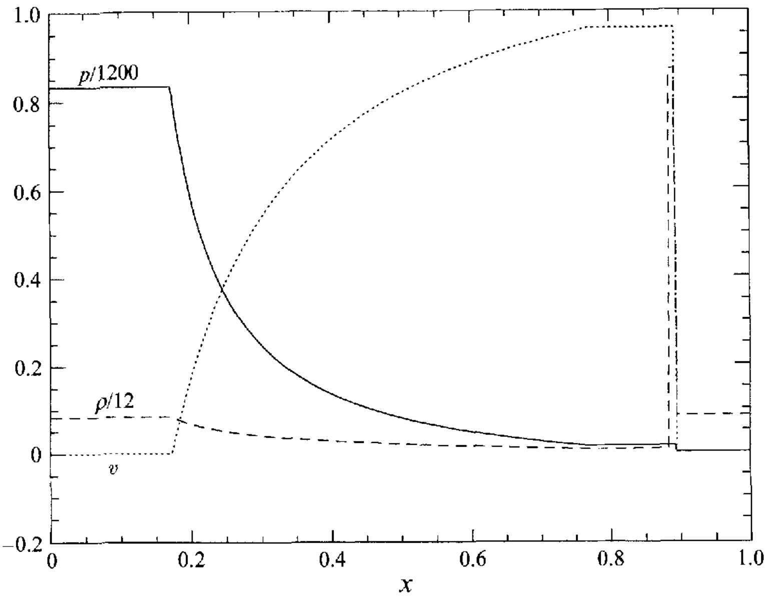

FIGURE 5. Analytical pressure, density and flow velocity profiles for the relativistic Riemann problem with initial data $p_L = 10^3, \rho_L = 1.0, v_L = 0; p_R = 10^{-2}, \rho_R = 1.0, v_R = 0$. A polytropic equation of state with $\gamma = \frac{5}{3}$ was assumed (case (*b*); see (4)). The initial discontinuity is located at $x = 0.5$. The solution is shown at $t = 0.4$.

(63) must be evaluated in state I_*, while (64) must be calculated with $a = I$ and $b = I_*$. Then, the mass flux across the shock is given by (61) and the shock velocity follows from (46).

In the case of rarefaction waves and for a polytropic EOS the isentropic character of the flow implies a relation between densities and pressures on both sides of the rarefaction, which reads

$$\frac{p_I}{p_{I_*}} = \left(\frac{\rho_I}{\rho_{I_*}}\right)^{\gamma}, \tag{71}$$

from which ρ_{I_*} can be determined. The velocities of the head and tail of the rarefaction, ξ_h and ξ_t, are given by the limiting values of ξ in (32):

$$\xi_h = \frac{v_I \mp c_{sI}}{1 \mp v_I c_{sI}} \tag{72}$$

and

$$\xi_t = \frac{v_{I_*} \mp c_{sI_*}}{1 \mp v_{I_*} c_{sI_*}}, \tag{73}$$

where c_{sI_*} is calculated from p_{I_*} and ρ_{I_*} using (34). The $-$ ($+$) sign applies to rarefaction waves propagating to the left (right). Finally, inside the rarefaction ($\xi_h < \xi < \xi_t$ for rarefactions propagating to the left; $\xi_t < \xi < \xi_h$ for those propagating to the right) the algebraic system defined by (31) and (38) must be solved with $a = I$ for the unknowns $c_s(\xi)$ and $v(\xi)$. The polytropic EOS can again be used to calculate $p(\xi)$ and $\rho(\xi)$ from $c_s(\xi)$.

This concludes the derivation of the analytical solution of the relativistic Riemann problem. It has been obtained by solving two algebraic implicit equations. These are (70), which allows one to compute p_*, and the equation derived by substituting $c_s(\xi)$

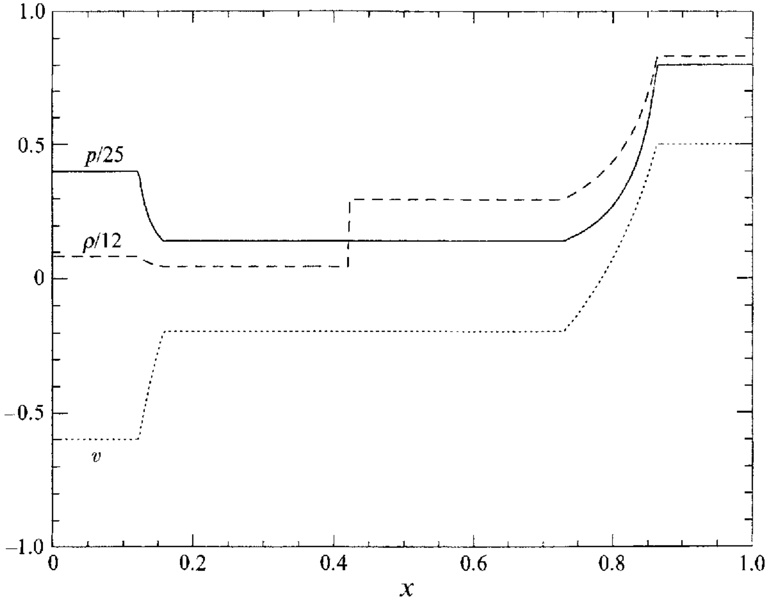

FIGURE 6. Same as figure 5, but for initial data $p_L = 10$, $\rho_L = 1.0$, $v_L = -0.6$; $p_R = 20$, $\rho_R = 10$, $v_R = 0.5$; $\gamma = \frac{5}{3}$ (case (*c*); see (5)).

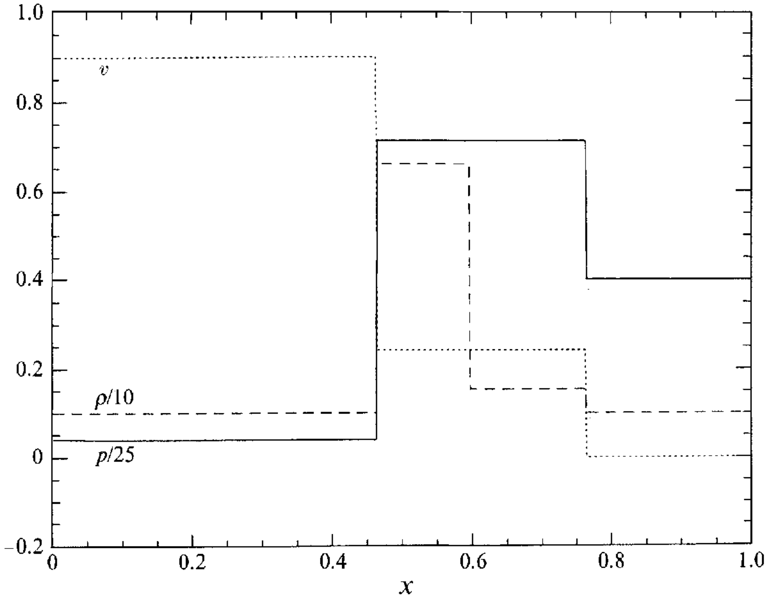

FIGURE 7. Same as figure 5, but for initial data $p_L = 1.0$, $\rho_L = 1.0$, $v_L = 0.9$; $p_R = 10$, $\rho_R = 1.0$, $v_R = 0$; $\gamma = \frac{4}{3}$ (case (*a*); see (3)).

or $v(\xi)$ in (38) by the expression given in (31). The latter equation is required to calculate the flow variables inside the rarefaction wave. Given the smoothness of the functions involved, one expects that numerical techniques for solving implicit equations (e.g. a Newton–Raphson iteration) should converge according to their theoretical convergence rates. Figures 5–7 show the analytical solution of three different Riemann problems corresponding to the types described in the Introduction (see (3)–(5)).

6. Application to numerical relativistic hydrodynamics

Relativistic hydrodynamics plays an important role in different fields of physics, e.g. in astrophysics, cosmology or nuclear physics. In extragalactic jets emanating from core-dominated radio sources associated with galaxies with active nuclei (see, e.g. Phinney 1987), or in current laboratory heavy-ion collisions (see, e.g. Strottman 1989) even ultra-relativistic flows are encountered. The necessity of simulating such relativistic flows which also involve strong shocks is triggering the development of relativistic hydro-codes.

As mentioned in the Introduction both approximate and exact Riemann solvers have become increasingly popular in thc development of high-resolution shock-capturing (HRSC) Newtonian hydro-codes. In the case of relativistic hydrodynamics HRSC techniques based on approximate Rieman solvers have recently been used to simulate one-dimensional flows (Martí, Ibáñez & Miralles 1991; Marquina *et al.* 1992; Schneider *et al.* 1993) and multi-dimensional ones (Font *et al.* 1993; Eulderink & Mellema 1993).

Using the analytical Riemann solution described above one can construct a relativistic version of Godunov's (1959) method, which is qualitatively as simple as its Newtonian analogue. In fact, we have obtained some preliminary results with a one-dimensional HRSC code based on this exact solution. The code has successfully passed the standard tests including the relativistic version of Sod's shock tube problem (Sod 1978), the relativistic shock reflection problem (Centrella & Wilson 1984), the propagation of a relativistic blast wave (Norman & Winkler 1986) and the relativistic version of the interaction of two blast waves (Woodward & Colella 1984). The results will be described in more detail in a forthcoming publication by the present authors.

The application of exact Riemann solvers in multi-dimensional calculations of relativistic flows unfortunately is more complicated than in Newtonian simulations. In order to explain this statement let us first reconsider some basic aspects of Newtonian simulations. Most modern multi-dimensional (explicit) finite-difference codes are based on the technique of directional splitting. In this technique the multi-dimensional problem is split into (two or three) sets of one-dimensional problems (the so-called sweeps along coordinate directions), which are solved consecutively in each timestep (see e.g. LeVeque 1991). Thus, a timestep consists of solving sets of one-dimensional Riemann problems in the presence of a tangential velocity, v^T (see figure 8).

As is well known (see e.g. Landau & Lifshitz 1987) in classical hydrodynamics, the decay of an initial discontinuity does not depend on the tangential velocity and, furthermore, this velocity is constant across shock waves and rarefactions. These facts reduce the problem to the solution of the one-dimensional problem plus the determination of the correct value of the tangential velocity in the intermediate states. The latter is given by

$$v^T_{I_*} = v^T_I, \tag{74}$$

where I stands for either L or R. Thus, as far as the Riemann solver is concerned, the extension of one-dimensional Newtonian hydro-codes to multi-dimensional ones is straightforward.

In relativistic calculations, however, one has to deal with the solution of Riemann problems in which the two components of the flow velocity (normal and tangential) are coupled owing to the presence of the Lorentz factor in the equations. A derivation similar to that described in §§3 and 4, but including a tangential component of the flow velocity in the initial condition, shows that for shock waves the presence of the tangential velocity component only complicates the algebraic expressions. In the case

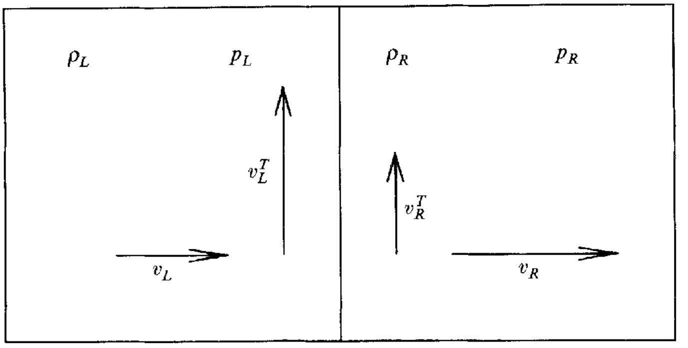

FIGURE 8. Schematic representation of two adjacent computational cells of a two-dimensional grid, which illustrates that for the solution of the Riemann problem in two-dimensional flows both the normal *and* the tangential components of the flow velocity must be taken into account on both sides of the interface.

of rarefactions, however, it requires the solution of a system of two ordinary differential equations (ODE). One ODE relates the variations of density and normal velocity inside the wave. The other ODE relates the normal and tangential component of the flow velocity. Although the need for an ODE solver makes the exact solution unattractive for modern HRSC codes, the solution still remains interesting, because it allows one to construct analytical solutions for multi-dimensional relativistic Riemann problems.

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Appendix. Shock velocity as a function of the mass flux across the shock

In this Appendix we discuss the solution of (46):

$$j = W_s \rho_a W_a (V_s - v_a), \tag{A 1}$$

which relates the shock velocity V_s with the mass flux j across the shock. All other quantities in (A 1) are known, because they describe the given pre-shock state.

Squaring (A 1) and rearranging the resulting terms one gets

$$(\rho_a^2 W_a^2 + j^2) V_s^2 - 2\rho_a^2 W_a^2 v_a V_s + \rho_a^2 W_a^2 v_a^2 - j^2 = 0, \tag{A 2}$$

which is a quadratic equation for the shock velocity in terms of the mass flux. The two solutions of (A 2) are

$$V_s^{\pm} = \frac{\rho_a^2 W_a^2 v_a \pm j^2 [1 + (\rho_a/j)^2]^{\frac{1}{2}}}{\rho_a^2 W_a^2 + j^2}. \tag{A 3}$$

In order to eliminate the unphysical of the two solutions let us first consider a shock propagating to the left ($j < 0$). According to (A 1) in this case

$$v_a > V_s. \tag{A 4}$$

Substituting V_s in (A 4) by the expression given in (A 3) one finds after some algebra

$$v_a > \pm [1 + (\rho_a/j)^2]^{\frac{1}{2}}. \tag{A 5}$$

In case of the plus sign the right-hand side of (A 5) is larger than 1, i.e. v_a is larger than the speed of light. Thus, V_s^+ obviously is an unphysical solution, and can be discarded for shocks propagating to the left. A similar reasoning shows that V_s^- is an unphysical solution for shocks propagating to the right ($j > 0$).

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